Origins
of the Calculus of Variations

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Origins of Mechanics

- **Archimedes**’ famous results
- **Nicole Oresme** (1348 – 1361)
  in the **College of Navarre** in Paris
- **Merton College**, Oxford

- **Galileo Galilei**
- **Pierre de Fermat**
- **Isaac Newton**

Archimedes' Results

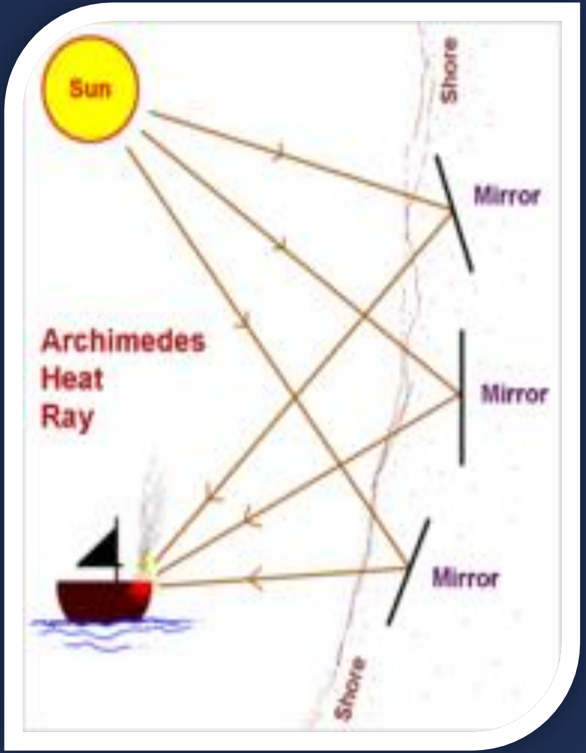
- **Archimedes** (287 BC – 212 BC)
- elements of statics

  (weight balance on a pulley =

  = moments equality)
- elements of hydrostatics
- On Method (known from 1906)

  – calculations of volumes, areas

  - early nontrivial results in calculus



Nicole Oresme (1323 – 1382)

- lectured in years 1348 – 1361 **at College of Navarre in Paris**, later he lived in Rouen
- **od 1377** – bishop in Lisieux
- **translated more Aristotle's papers.**
- He was against astrology and prophecy (but he believed in magic).
- Dissapproved devaluation of coins by governments, economy
- many papers from astronomy and mechanics, musics.

Oresme’s work

Transactions

- *Tractatus proportionum* (about 1350)
- *Algorismus proportionum* (printed in 19th century, but in Oresme’s time was manuscript known)
- *On configurations of qualities*

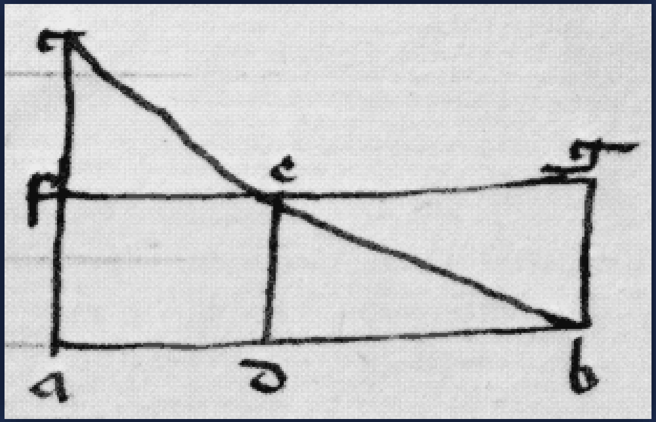
  (De configuratio)
- *Tractatus on creating of forces and measure unequality*
- (before year 1371)

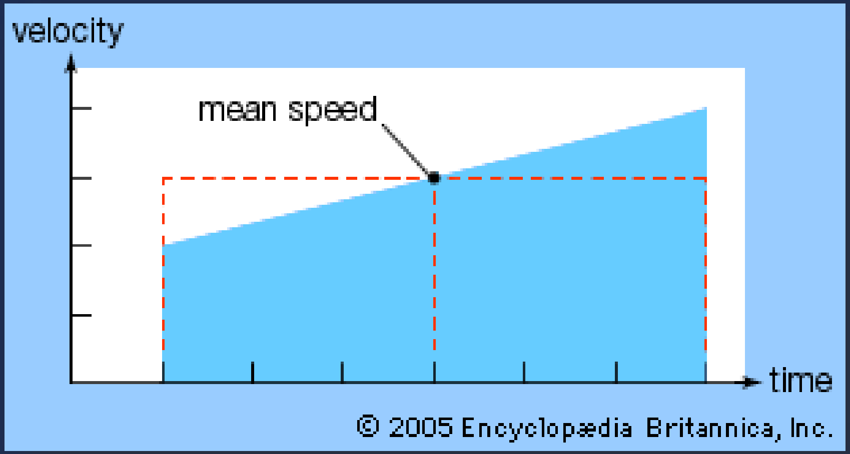
What yields new?

- Used geometric expressions of quantities and its interdepency.
- Used coordinates, possibilty of geometric representation of functions.
- **Velocity is a time function.**

Merton Acceleration Theorem

- **1330 – Merton College, Oxford**
- **Distance an object moves under uniform acceleration is equal to the width of the time interval multiply by velocity at the midpoint of the interval, its mean speed.**
- **time x velocity, constant acceleration**
- **1361 Oresme – geometrical proof**





Oxford, Merton College

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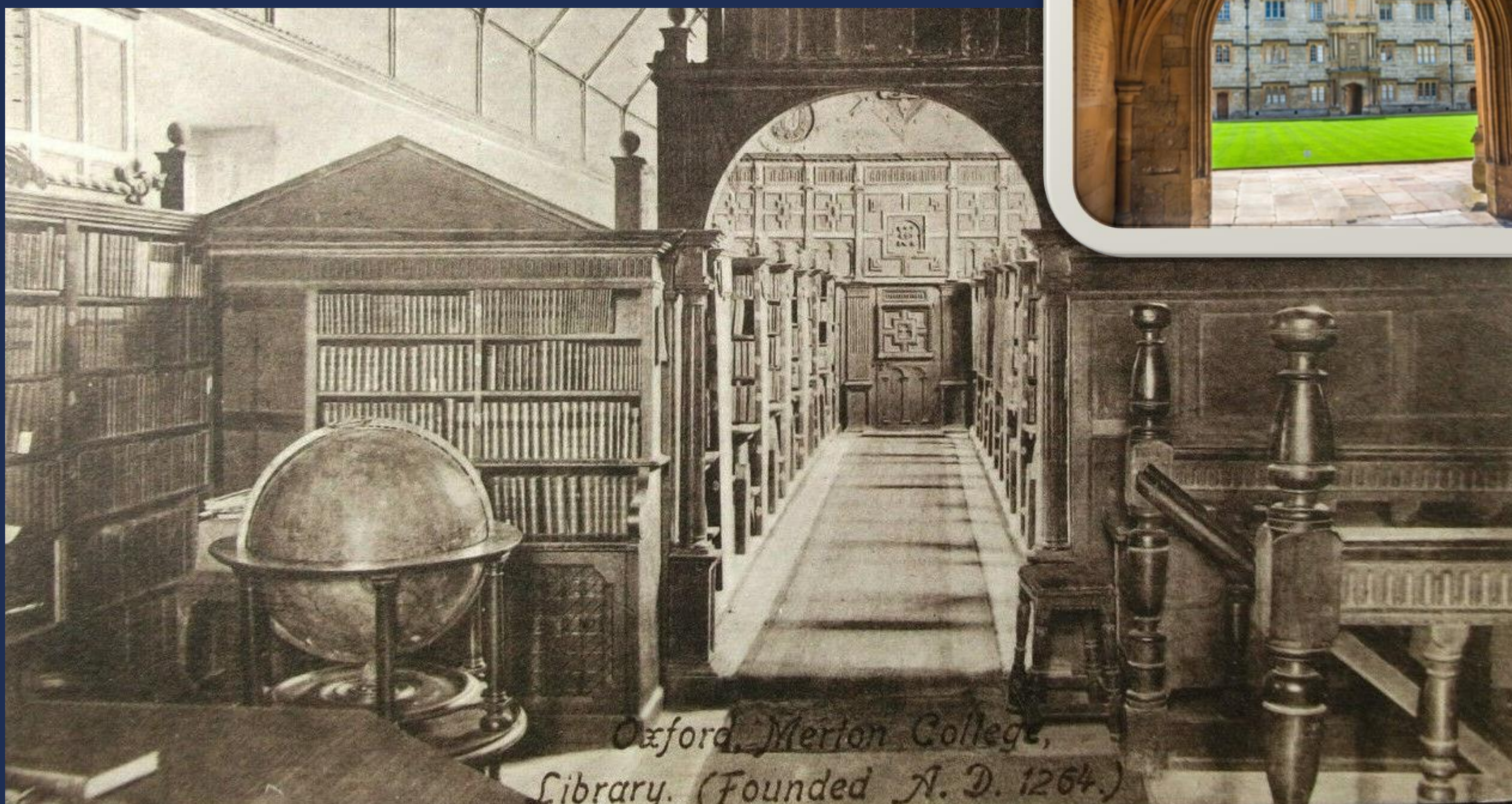


Oxford Calculators

- Thomas Bradwardine (1300 – 1349)

  Arcibishop in Canterbury

- William Heytesbury (cca 1300 – 1372/73)

  Chancellor of the Oxford University

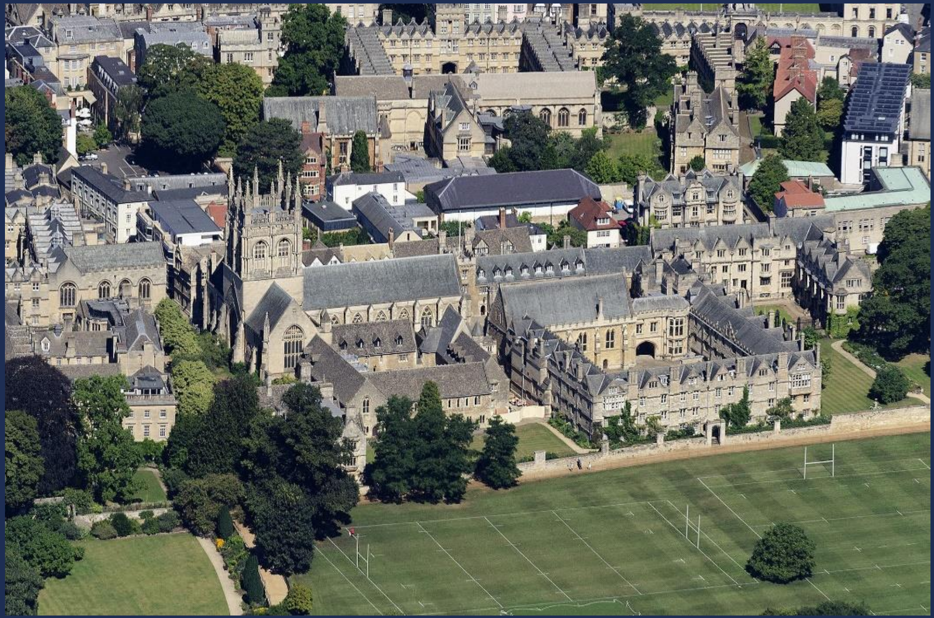
- Richard Swineshead

  Liber calculationum nickname: Calculator

Oxford, Merton College

MC was founded -1264.

Oxford Calculators, 14th Century



Galileo Galilei (1564 – 1642)

- **In the year 1604 in the letter - about dependence of movement to *t***
- **Originally thought relation of speed to time $v = k.t$**

  and *relation of speed to* distance

  $$v = k.s$$
- **not until in the year 1638.**

  He desided again

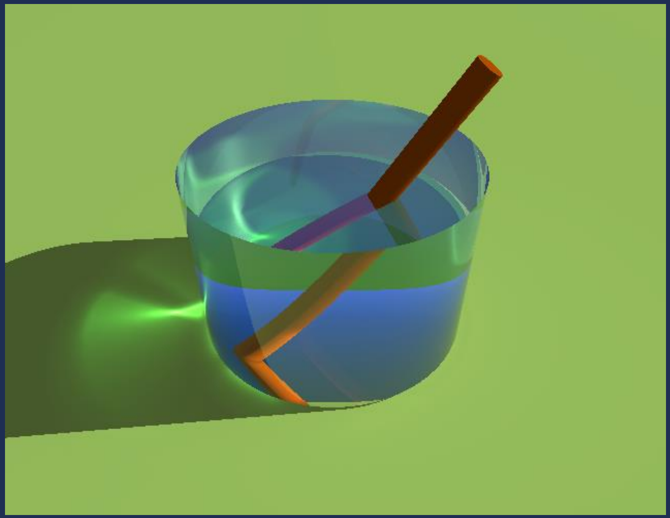
  for $v = k.t$
- **Then derived trajectory of projectile**
- **The principle of inertia**
- **He interested in resolution of forces**

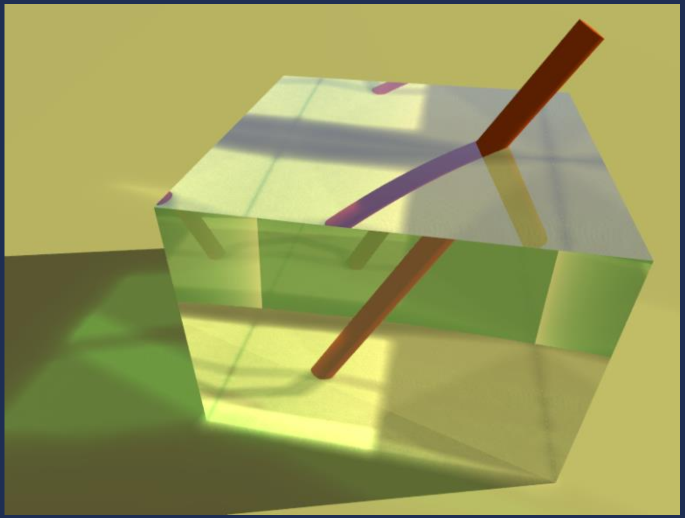
→ **Stevin, Roberval (1636)**

Pierre de Fermat (1601-1667)

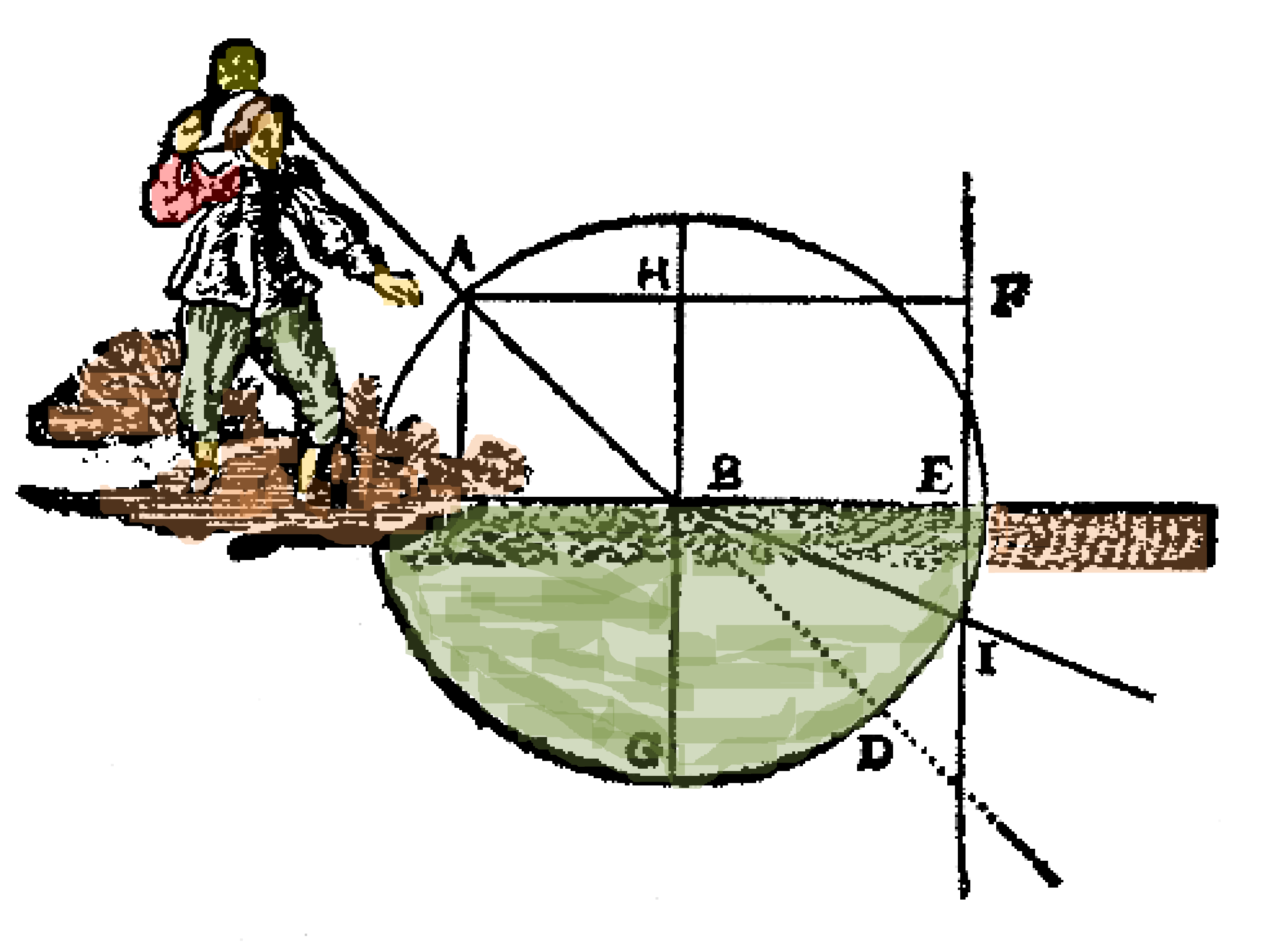
Fermat's principle → $$\frac{\sin i}{\sin r} = \frac{v_1}{v_2}$$

$$J = \int_{\sigma_1}^{\sigma_2} \frac{ds}{v} = \int_{\sigma_1}^{\sigma_2} \frac{n}{c} d\sigma = \frac{1}{c} \int_{\sigma_1}^{\sigma_2} n(x, y.z) \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, d\sigma$$

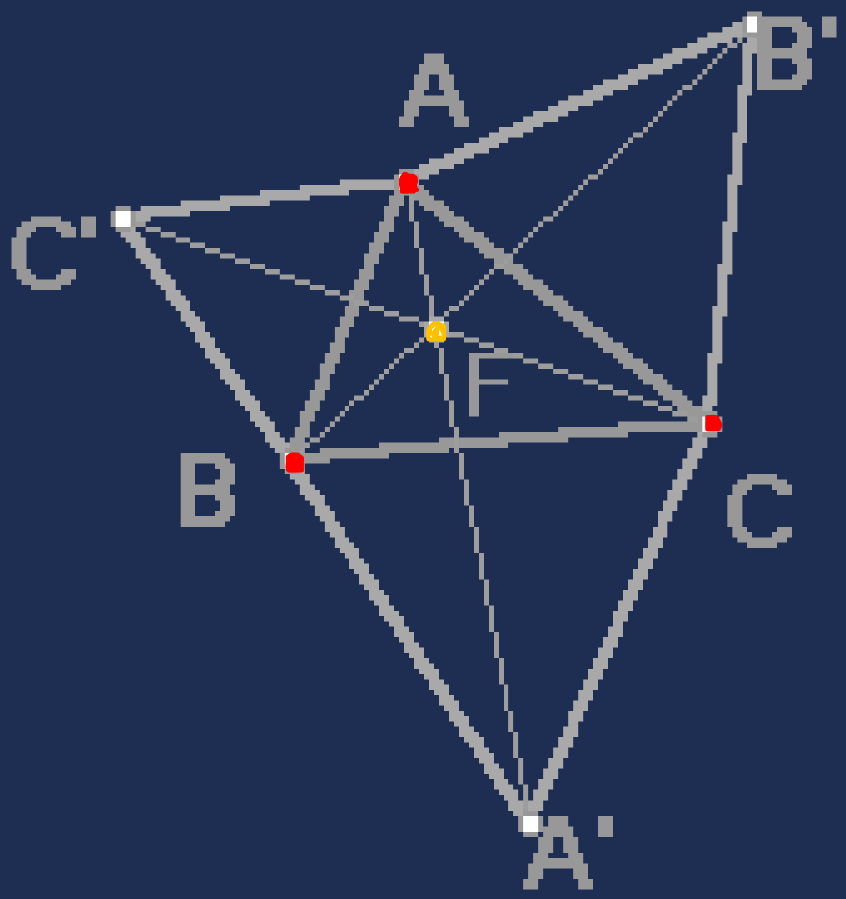


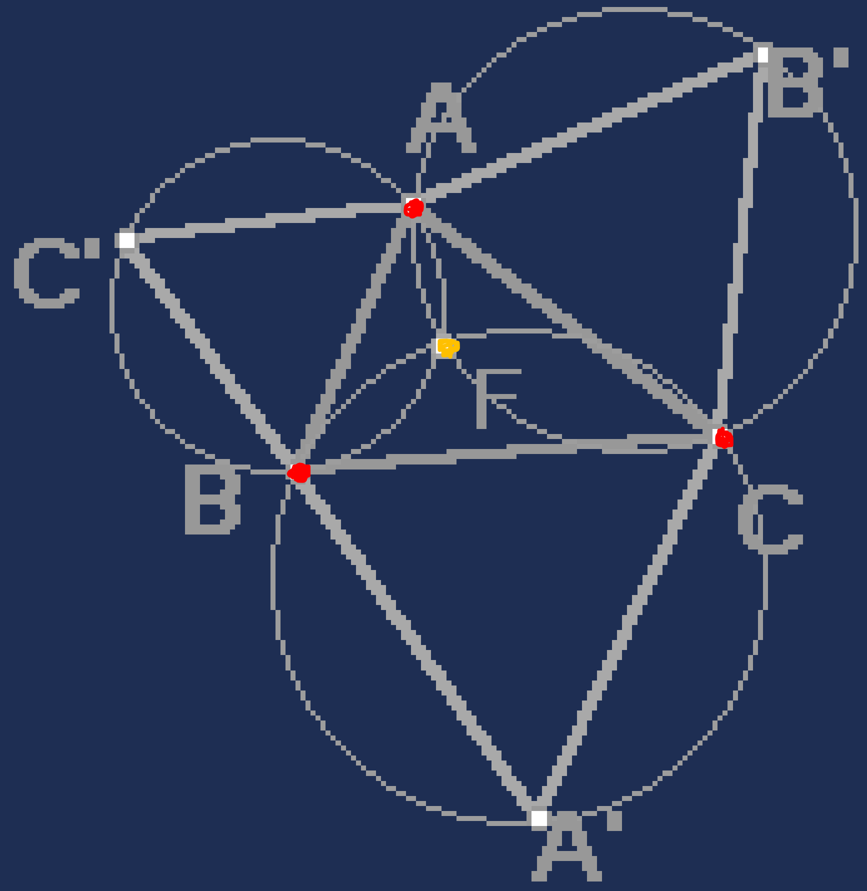


Descartes investigations



Fermat or Toricelli point





Celestial Mechanics

- Johannes Kepler (1571 -1630)

  Astronomia nova, 1609
- Isaac Newton (1642 -1727)
- Edmond Halley (1656 -1742)
- Pierre Simon Laplace (1740-1827)

Mechanical Curves

René Descartes (1596–1650): *La Géometrie*

- **geometric** today **algebraic** curves
- **mechanical** today **transcendental**

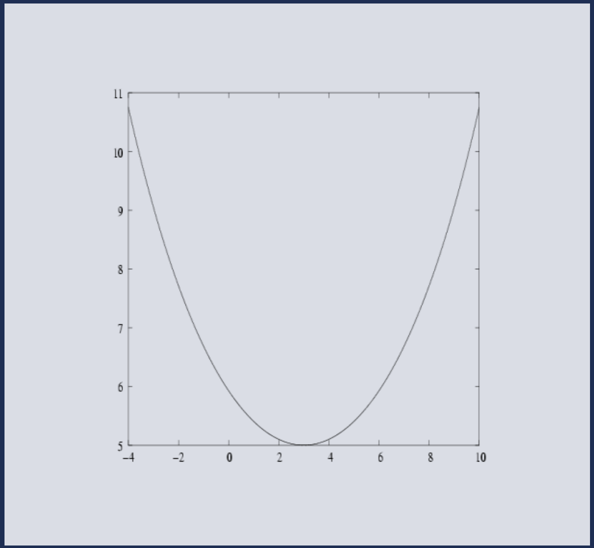
Why mechanical?

Ancient Greeks defined the with help of certain hypotetic mechanism.

Example: epicycles

(with help of movement of one circle around the second one)

Next Examples



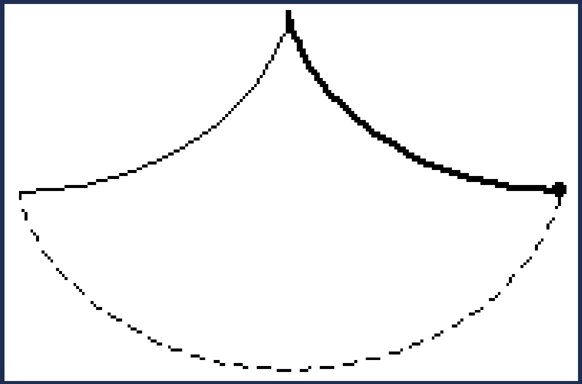
- **Catenary** (chain curve)

$$y(x) = A\cosh\frac{x-B}{A}$$

- **Cykloid** – the point moving at the circumference of the circle moving at the plane.
- **Blaise Pascal** described properties of cykloids in the year 1638

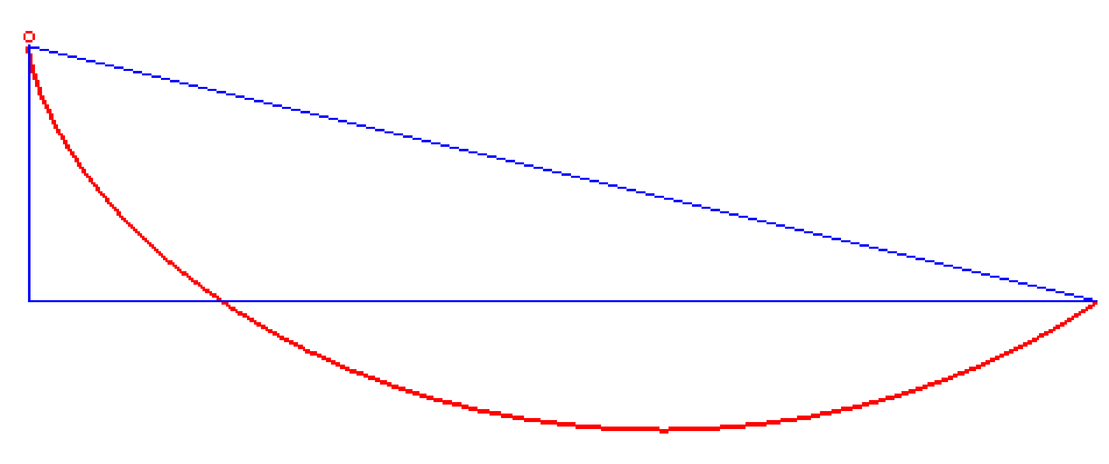
  in the *paper On cycloid.*

Tautochrone



- „Isochronic curve”
- 1659 – Christian Huygens in 17 years
- 1673 – He used of geometric properties for the construction of pendulum clock.
- A period cykloidal pendulum is independent on amplitud.

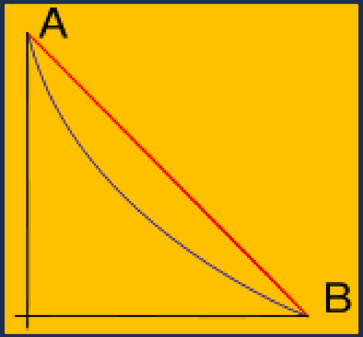
Brachistochrone

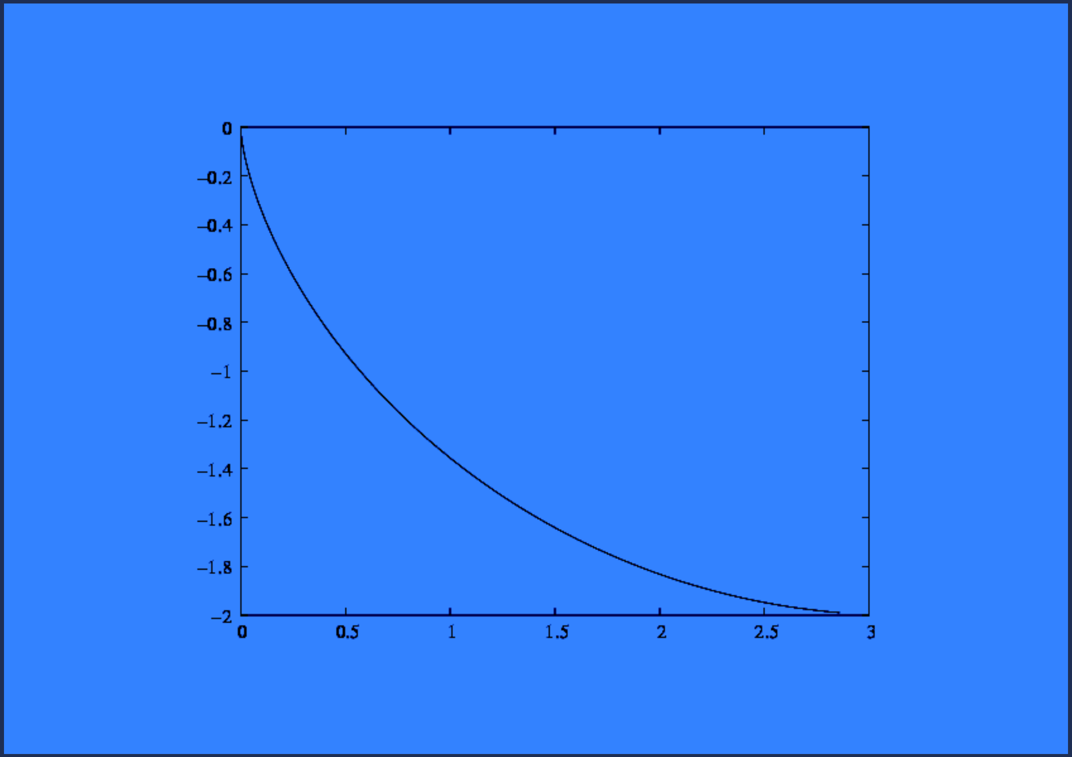


- **„Curve of the shortest time”**
- The formulation of problem - **Jacob Bernoulli**
- 1697 – Johann Bernoulli, Leibniz, l'Hôpital, Newton, Jacob Bernoulli
- **Jacob Bernoulli – „variable curve”**
- One of the first tasks of calculus of variations!

Brachistochrone II

$$x(t) = C(t - \sin t)$$
$$y(t) = C(1 - \cos t)$$





18th century

Geodetics – a trajectory of minimal length at the plane.

- Efforts to find the shortest ways at the Earth surface,
  the form of it was not known.
- The hypothesis of mathematicians – the Earth has the form of rotational elipsoid - later spheroid.
- Alexis C. **Clairaut**, Friedrich **Helmert** – deformation **1728 – Johann Bernoulli**

- The suggestion to **Leonhard Euler** to solve of a problem of finding geodetics at the surface using of osculating planes of geodetics
- Leonhard Euler founded calculus of variations solving of this problem.
- Comm. Acad. Sci. Petrop., 3, 1728, 110 – 124, publ. 1732

Pierre-Louis Moreau de Maupertuis
1698-1759

He took up Fermat.

- **1744 - Principle of minimal action**
- **First universal law of nature**
- **A proof of existence of God**
- **Euler in addition, where**
  he studied motion of particles at plane curve, he supposed, that the velocity is dependence at the position of particle.

- **Maupertuis** $mvs = \min.$ **Euler** $\partial \int v \, ds = 0$

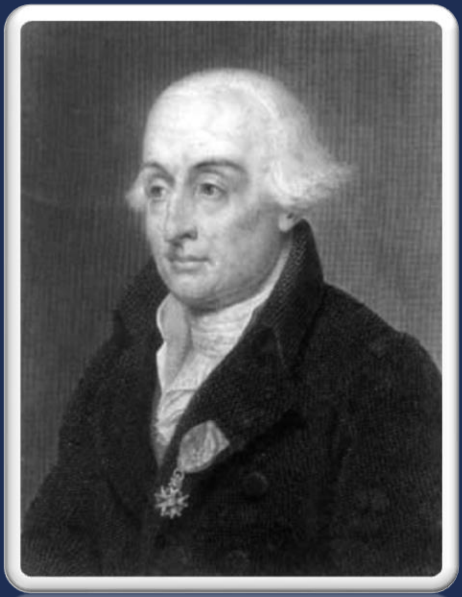
Euler a Lagrange

- 1734 – Euler generalised of the problem of brachistochrone by minimalization of other quantities than time.

Leonhard Euler

- 1750 - Joseph Louis Lagrange
  In 19 years old he was inspirated by Euler.
- He found pure analytical methods, 1755 – the letter to Euler with their description
- 1756 Euler published Lagrange's letter in Berlin, where method named calculus of variations.

Formulation of the problem



- **Basic task – minimalization or maximalization of the integral**

$$J(y(x)) = \int_{x_1}^{x_2} f(x, y, y')\, dx$$

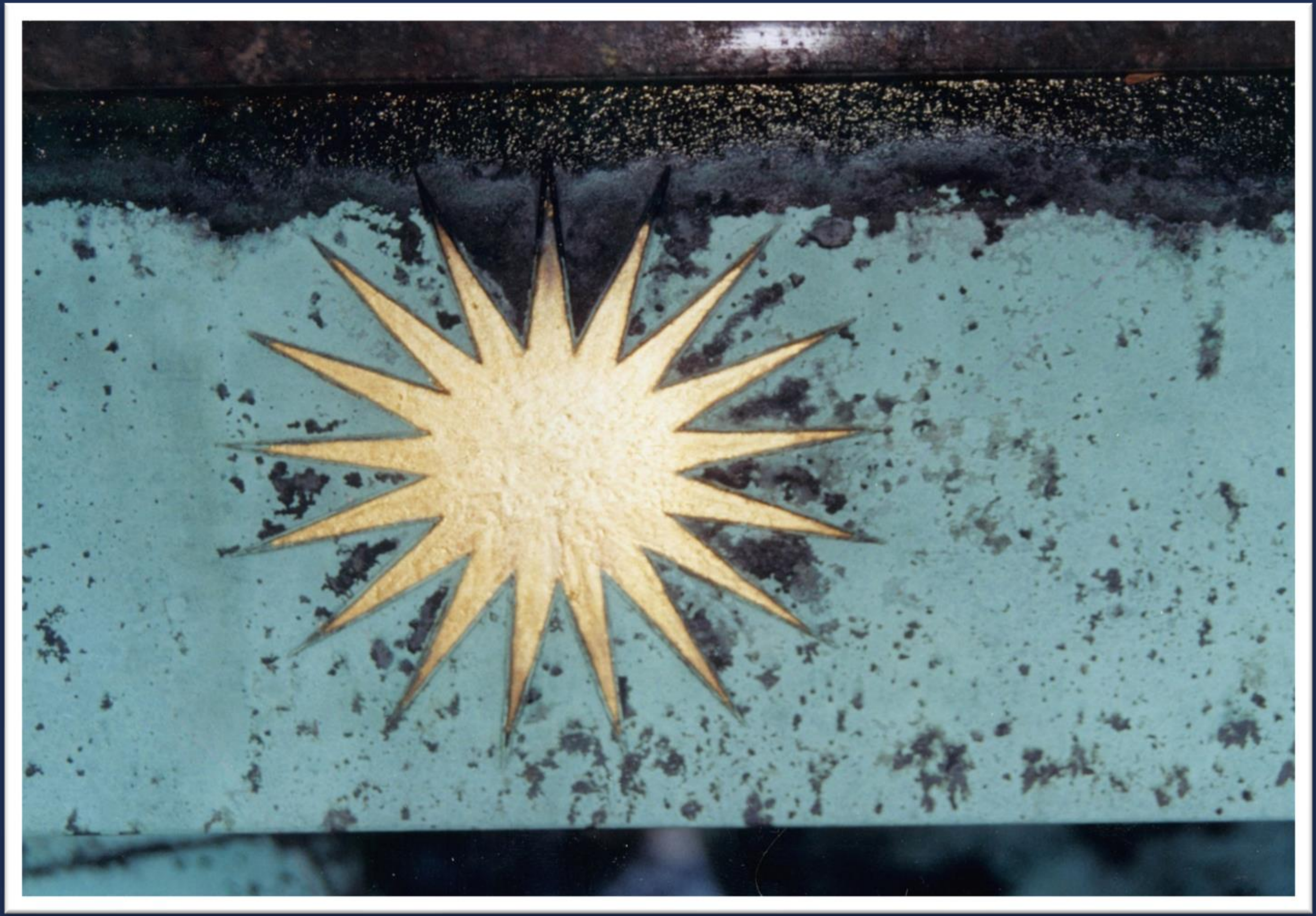
- **1762 – Lagrange – *Essai d' une nouvelle méthode pour déterminer les maxima et les minima des formules intégrales in définiés.***

Gauss' name

- Gauss elimination method in the matrix theory
- Gauss curve and normal law (distribution)
  in probability and statistics,
  in financial science, in geodesy, physics
- Unit "gauss" in magnetism
- Gauss method for calculations of Eastern
- Gauss plane, Gauss integers
- Gauss quadrature
- Gauss transformation, Gauss curvature,
  etc.

Carl Friedrich Gauss, Braunschweig

Heptadecagon is left in the stand.



The statue of Gauss, Braunschweig

The detail of heptadecagon

Inter Iovem et Martem interposui planetam

- **Johannes Kepler (1571-1630), hypothesis**
- **Bonnet's row - 1772**

  distance of k-th satelite $4 + 3.2^{k-2}$
- **Wurm's row - 1787** $387 + 293.\ 2^{k-2}$

- **Professor Studnička** ***- a hypothesis on existence of a planet between Mars a Jupiter***
- **Organization of this search**
- **1796 Lalande a 1799 Olbers.**

Asteroid Ceres

- **January 1st 1800 Piazzi in Palermo was successful.**

  He found a little solid of solar system Ceres, but it was missed very soon.

- **Studnička: „ *Dr. Gauss published briefly, but very exact description of its orbit. He had for calculation three observations of Piazzi from January 2nd and 22nd, then from February 11th*".**

  „The calculation was made with help of the new method and quite exact, so Franz von Zach in December 7th of this year had found missed object in the orbit calculated by Gauss and also Heinrich Olbers was successful in January 1st 1801 ."

- **„Gauss calculation showed one searched particle of sand at seashore."**

Calculations of orbits of asteroids: Ceres

- **6 observations**,
  when the asteroid was in opposition
  and when it was the most near to the Earth.
- **12 equations with 6 unknowns**
  (middle anomaly, middle daily motion,
  the length of perihelium, excentricity,
  the length increasing node, inclination).

- After getting approximate solution he
  linearized the system of 12 equations,
  he did not used 10th one (not exact).

GEM – the Method for Ceres

- He used 11 equations,
  from them
  derived 6 normal equations for 6 corrections,

- he used for solution
  of the system of equations

Gauss' Elimination Method.

Calculation of orbits of asteroids: Pallas

- Unknowns in the system are again corrections to approximate solution.
- He used GEM and the transformation of quadratic form to diagonal quadratic, weight sum of squares Ω.
- **He minimalized the sum Ω.**
- 1801 – firstly used the **Least Squares Method** (Ceres).
- 1810 – explication of method (Pallas).

The most important papers

Disquisitio de Elementis Ellipticis Palladis …, Göttingen 1810.

- Disquisitiones arithmeticae, 1801
- Theoria motus corporum coelestium in sectionibus conicis Solem ambientium,1809
- Disquitiones generales circa superficies curvas,1827

„Mathematicians applauded Gauss, but they did not understand him!“

Calculus of Variations at the Prague's Technical University and Prague's University

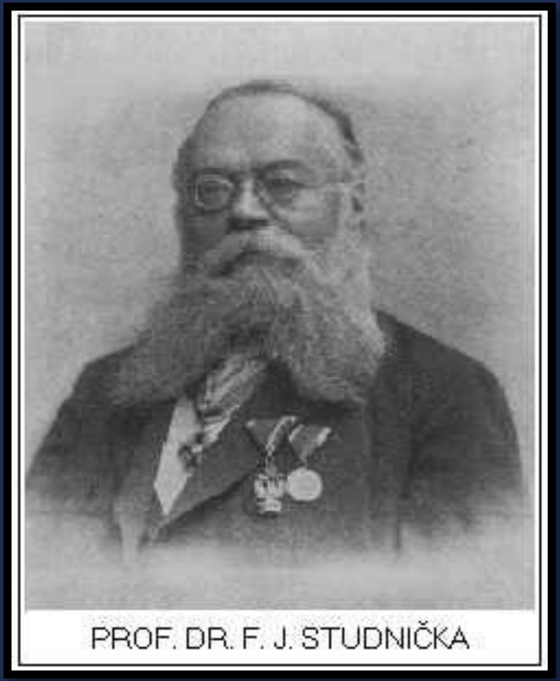
František Josef Studnička (1836 – 1903)

- **1864 –** FJS at the polytechnics
- 1865/66

  Differencial equations and calculus of variations 5 0
- **1871**
- Lecture on the origin and development of calculus of variations, first lecture at Prague's University, 1871, 15 pages
- **1872**

  On the Calculus of Variations, 54 pages

Who was FJS?



- *„He overpowered himself by rows of ciphers mysteries of world and life“ ,* Gold Prague.
- *„Logarithms presented by Studnička are more clear than a light of candle“* - thirty years of 20th cent.
- *„Who was not a star in calculations, he liked Studnička's hearty Czech lectures from geography, astronomy, or in meteorology“,* Gold Prague.